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Stress and tangent update equations for combined time-hardening creep and J_2 plasticity in an implicit hypo-elastic formulation

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Abstract

This work is motivated by the need to analyze the behavior of metallic nuclear fuels which under normal operating conditions build up stresses due to non-homogeneous thermal expansion, fission gas and solid product swelling among other phenomena that are simultaneously relaxed by creep and plastic flow. This report details the stress and tangent update equations for combined J_2 based rate independent plasticity and time-hardening creep effects in a fully implicit hypo-elastic formulation involving two cases: pure creep without plasticity where the yield criterion has not yet been met and the combined effect of both creep and plasticity beyond yield. Closed form expressions for the consistent material tangent to be used in both cases are derived which can be used in implicit codes and is expected to help in obtaining optimal convergence rates.

1. Background and motivation

Metallic nuclear fuels exhibit extremely complex behavior during irradiation which include but are not limited to thermal expansion, swelling due to fission and solid products of nuclear fission, phase transformation, creep and plastic flow, damage and cracking. Over the years, many numerical codes have been developed to simulate metallic nuclear fuels, most of which are based on reduced order models [1]. As there is increasing need to perform full scale 3D analyses over long periods of operation (time scales of weeks to months), there is a need to develop full 3D equations for the material behavior [2]. Moreover to achieve this, implicit finite element codes are being used increasingly so that there is no restriction on the time step as is common in explicit finite element codes. To ensure optimal convergence rates of the global finite element equations, the consistent material tangent should be provided to the finite element solver in accordance with the algorithm used for the stress update. A material tangent evaluated numerically could be used but would also be expensive to compute [3, 4]. A closed form expression for the material tangent would be considerably more computationally efficient than using an explicit finite element solution or evaluating a numerical tangent.

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Metallic fuels often exhibit creep and plastic flow, especially when operating at high temperatures which relax stresses in the fuel built up by other phenomena such as thermal expansion and fission gas swelling. In this work, closed form expressions are derived for the stress and consistent material tangent update for either of two possible cases where the material: (1) is within the yield surface and undergoing creep flow or (2) has yielded and undergoing both creep and plastic flow simultaneously. Although metallic fuel alloys such as U-Pu-Zr are often assumed to be elastic-perfectly plastic [1], it is assumed for the present purposes that the plastic flow follows an isotropic hardening law to introduce more generality in the constitutive behavior.

2. Evolution of stress

Assume that the total strain tensor $\boldsymbol{\varepsilon}$ can be additively decomposed into a linear elastic strain $\boldsymbol{\varepsilon}^{el}$, thermal strain $\boldsymbol{\varepsilon}^{th}$, plastic strain $\boldsymbol{e}^{pl}$ and creep strain $\boldsymbol{e}^{cr}$ such that in rate form,

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}^{el}} + \dot{\boldsymbol{\varepsilon}^{th}} + \dot{\boldsymbol{\varepsilon}^{pl}} + \dot{\boldsymbol{\varepsilon}^{cr}} \tag{1}$$

Assume that the material response is linear elastic such that,

$$\dot{\boldsymbol{\sigma}} = \mathbb{C} : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}^{th}} - \dot{\boldsymbol{\varepsilon}^{pl}} - \dot{\boldsymbol{\varepsilon}^{cr}}\right) \tag{2}$$

Using a backward (implicit) Euler integration scheme (Eqns. (1.4.4), Simo and Hughes [5]) and assuming $\mathbb{C}$ to be constant, the incremental form of (2) can be written as,

$$\Delta\boldsymbol{\sigma} = \mathbb{C} : \left(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{th} - \Delta\boldsymbol{\varepsilon}^{pl} - \Delta\boldsymbol{\varepsilon}^{cr}\right)_{n+1} \tag{3}$$

which can be written in terms of the n^{th} and $n+1^{th}$ time step as,

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_n + \mathbb{C} : \left(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{th} - \Delta\boldsymbol{\varepsilon}^{pl} - \Delta\boldsymbol{\varepsilon}^{cr}\right)_{n+1} \tag{4}$$

where the right hand side has inelastic strains which may be functions of variables that are evaluated at at time step $n+1$. As is commonly done, the thermal strain is assumed to be purely volumetric in nature and the creep and plastic strains are assumed to be purely deviatoric such that the volumetric and deviatoric components of the elastic strain can be written as,

$$\mathrm{vol}(\Delta\boldsymbol{\varepsilon}^{el}) = \mathrm{vol}(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{th}) = \frac{1}{3}\mathrm{tr}(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{th})\mathbf{1} \tag{5}$$

$$\mathrm{dev}(\Delta\boldsymbol{\varepsilon}^{el}) = \Delta\boldsymbol{e} - \Delta\boldsymbol{e}^{pl} - \Delta\boldsymbol{e}^{cr} \tag{6}$$

where vol$(\cdot)$ and dev$(\cdot)$ denote the volumetric and deviatoric components of their arguments, $\mathbf{1}$ is the second order identity tensor or the Kronecker delta tensor and $\Delta\boldsymbol{e}$ is the deviatoric part of the total strain increment. Rearranging terms in the (4), and writing the current stress $\boldsymbol{\sigma}_{n+1}$ in terms of the hydrostatic and deviatoric components,

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}^h_n + \Delta\boldsymbol{\sigma}^h_{n+1}\left[\mathrm{vol}(\Delta\boldsymbol{\varepsilon}^{el}_{n+1})\right] + \boldsymbol{s}_n + \Delta\boldsymbol{s}_{n+1}\left[\mathrm{dev}(\Delta\boldsymbol{\varepsilon}^{el}_{n+1})\right] \tag{7}$$

in which the hydrostatic component of the stress at $n+1$ can be completely determined from knowing the stress state at n, and the total strain and thermal strain increments.

The deviatoric component of stress at $n+1$ is written as,

$$\begin{aligned} \boldsymbol{s}_{n+1} &= \boldsymbol{s}_n + \Delta \boldsymbol{s}_{n+1}\left[\mathrm{dev}(\Delta\boldsymbol{\varepsilon}^{el}_{n+1})\right] && (8)\\ &= \boldsymbol{s}_n + \Delta \boldsymbol{s}_{n+1}\left(\Delta\boldsymbol{e}_{n+1} - \Delta\boldsymbol{e}^{pl}_{n+1} - \Delta\boldsymbol{e}^{cr}_{n+1}\right) && (9)\\ &= \boldsymbol{s}_n + 2\mu\left(\Delta\boldsymbol{e}_{n+1} - \Delta\boldsymbol{e}^{pl}_{n+1} - \Delta\boldsymbol{e}^{cr}_{n+1}\right) && (10)\end{aligned}$$

Assuming that the step from n to $n+1$ is a purely elastic one, and writing the resulting stress as s^{trial}_{n+1},

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}^{trial}_{n+1} - 2\mu\left(\Delta\boldsymbol{e}^{pl}_{n+1} + \Delta\boldsymbol{e}^{cr}_{n+1}\right) \tag{11}$$

where the trial stress is $\boldsymbol{s}^{trial}_{n+1} = \boldsymbol{s}_n + 2\mu\left(\Delta\boldsymbol{e}_{n+1}\right)$, and the plastic and creep strains at $n+1$ are yet to be determined.

2.1. Rate independent plasticity

Under the assumption of classical rate independent associative plasticity, the plastic strain rate is written as

$$\dot{\boldsymbol{e}}^{pl} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{12}$$

where $\dot{\lambda}$ is a positive scalar quantity and f is a specified yield function generally written in terms of the deviatoric stress $\boldsymbol{s}$. The incremental form for the plastic strain can be written as,

$$\Delta\boldsymbol{e}^{pl}_{n+1} = \left(\Delta\lambda\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)_{n+1} \tag{13}$$

In J_2 plasticity, the yield function is in general is written in terms of the second invariant of the deviatoric stress,

$$J_2 = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s} \tag{14}$$

where $\boldsymbol{s}$ is the deviatoric stress defined by by $\boldsymbol{s} = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{1}$. A common form for the yield function f is,

$$f = \sqrt{2J_2} - \sqrt{\frac{2}{3}}\sigma_Y(\alpha) \tag{15}$$

where $\sqrt{2J_2}$ amounts to the magnitude of the deviatoric stress, $||\boldsymbol{s}|| = \sqrt{\boldsymbol{s} : \boldsymbol{s}}$, σ_Y is the yield stress of the material which can be a function of the accumulated plastic strain α whose evolution is assumed to be,

$$\dot{\alpha} = \dot{\lambda}\sqrt{\frac{2}{3}} \tag{16}$$

the algorithmic counter part of the yield function, written at time $n+1$ is,

$$f_{n+1} = ||\boldsymbol{s}_{n+1}|| - \sqrt{\frac{2}{3}}\sigma_Y(\alpha_{n+1}) \tag{17}$$

where the accumulated plastic strain is given by,

$$\alpha_{n+1} = \alpha_n + \Delta\lambda \sqrt{\frac{2}{3}} \tag{18}$$

Knowing that the second term in the yield function in (17) is a constant yield stress at $n + 1$, the gradient of f with respect to the stress is simply,

$$\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)_{n+1} = \frac{\partial \|\boldsymbol{s}_{n+1}\|}{\partial \boldsymbol{\sigma}_{n+1}} \tag{19}$$

$$= \frac{\boldsymbol{s}_{n+1}}{\|\boldsymbol{s}_{\boldsymbol{n+1}}\|} \tag{20}$$

$$= \boldsymbol{n}_{n+1} \tag{21}$$

Therefore, substituting (21) in (13), the plastic strain increment can be written as,

$$\Delta \boldsymbol{e}^{pl}_{n+1} = \Delta\lambda \boldsymbol{n}_{n+1} \tag{22}$$

where $\Delta\lambda$ specifies that magnitude of plastic strain increment and $\boldsymbol{n}_{n+1}$ specifies the direction of plastic flow.

2.2. *Time-hardening creep*

Time dependent creep laws are often written in a strain-hardening or time-hardening form. A law of the latter type is assumed here, where the creep strain rate is,

$$\dot{\boldsymbol{e}}^{cr}_{n+1} = \dot{\bar{e}}^{cr}_{n+1} (\tilde{\sigma}_{n+1}, T_{n+1}, t_{n+1}) \, \tilde{\boldsymbol{n}}_{n+1}(\boldsymbol{s}) \tag{23}$$

The creep strain increment can be written as,

$$\Delta \boldsymbol{e}^{cr}_{n+1} = \Delta \bar{e}^{cr}_{n+1} (\tilde{\sigma}_{n+1}, T_{n+1}, t_{n+1}) \, \tilde{\boldsymbol{n}}_{n+1}(\boldsymbol{s}) \tag{24}$$

where $\Delta \bar{e}^{cr}_{n+1}$ is the magnitude of the creep strain increment which depends on the current von Mises stress, time, temperature and other constants which are material parameters that can be estimated from a uniaxial creep test, and $\tilde{\boldsymbol{n}}_{n+1}(\boldsymbol{s})$ is the direction of flow of creep which is some function of the deviatoric stress. As is commonly done, the direction of creep is assumed to be such that,

$$\tilde{\boldsymbol{n}}_{n+1} = \frac{\partial \tilde{\sigma}_{n+1}}{\partial \boldsymbol{\sigma}_{n+1}} \tag{25}$$

where $\tilde{\sigma}_{n+1}$ is the von Mises stress. To be consistent with the direction of plastic flow, the direction of creep is written in terms of $\boldsymbol{n}$ such that,

$$\Delta \boldsymbol{e}^{cr}_{n+1} = \Delta \bar{e}^{cr}_{n+1} (\tilde{\sigma}_{n+1}, T_{n+1}, t_{n+1}) \sqrt{\frac{3}{2}} \boldsymbol{n}_{n+1} \tag{26}$$

Consider that a power law for creep strain is assumed in it's 'time hardening' form such that the creep strain rate is given by,

$$\dot{\bar{e}}^{cr} = A\tilde{\sigma}^m t^n e^{-Q/RT} \tag{27}$$

where $\tilde{\sigma}$ is the von-Mises stress, t is time, Q is the activation energy of the creep mechanism, R is the Boltzmann's constant, T is the absolute temperature and A,m and n are material parameters. Using an implicit time integration scheme, the creep strain increment can be written as,

$$\Delta\bar{e}^{cr}_{n+1} = \Delta t A\tilde{\sigma}^m_{n+1} t^n_{n+1} e^{-Q/RT_{n+1}} \tag{28}$$

where the material parameter n appearing as the exponential term for time t is not to be confused with the number of the time step appearing in the subscript.

2.3. Evaluating the deviatoric stress

Substituting (26), (28) and (22) in (11)

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}^{trial}_{n+1} - 2\mu\left[\Delta\lambda\boldsymbol{n}_{n+1} + \left(\Delta t A\tilde{\sigma}^m_{n+1} t^n_{n+1} e^{-Q/RT_{n+1}}\right)\sqrt{\frac{3}{2}}\boldsymbol{n}_{n+1}\right] \tag{29}$$

The unit normal tensor can be evaluated by knowing the trial stress state at $n+1$, which can be determined from the state of the material at n and the total applied strain increment ($\boldsymbol{s}^{trial}_{n+1} = \boldsymbol{s}_n + 2\mu\,\Delta\boldsymbol{e}_{n+1}$). Both the deviatoric and the trial deviatoric stress, $\boldsymbol{s}_{n+1}$ and $\boldsymbol{s}^{trial}_{n+1}$ can be shown to have the same unit normal tensor $\boldsymbol{n}_{n+1}$ [5], therefore can be determined as,

$$\boldsymbol{n}_{n+1} = \frac{\boldsymbol{s}^{trial}_{n+1}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} = \frac{\boldsymbol{s}_{n+1}}{\|\boldsymbol{s}_{n+1}\|} \tag{30}$$

Contracting with both sides of (29) with the unit normal tensor $\boldsymbol{n}_{n+1}$, one obtains,

$$\|\boldsymbol{s}_{n+1}\| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\,\Delta\lambda - 2\mu\,\sqrt{\frac{3}{2}}\left(\Delta t A\tilde{\sigma}^m_{n+1} t^n_{n+1} e^{-Q/RT_{n+1}}\right) \tag{31}$$

Writing (31) as a residual,

$$R = \|\boldsymbol{s}_{n+1}\| - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\,\Delta\lambda + 2\mu\,\sqrt{\frac{3}{2}}\left(\Delta t A\tilde{\sigma}^m_{n+1} t^n_{n+1} e^{-Q/RT_{n+1}}\right) \tag{32}$$

2.4. Pure creep

In the case of creep without any plastic effects, the nonlinear equation to be solved to evaluate the deviatoric stress at the current time step reduces to,

$$R = \|\boldsymbol{s}_{n+1}\| - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\,\sqrt{\frac{3}{2}}\left(\Delta t A\tilde{\sigma}^m_{n+1} t^n_{n+1} e^{-Q/RT_{n+1}}\right) \tag{33}$$

The von-Mises stress can be written in terms of the deviatoric stress as,

$$\tilde{\sigma}_{n+1} = \sqrt{\frac{3}{2}}\|s_{n+1}\| \tag{34}$$

Substituting into Eqn, (33),

$$R = \|\boldsymbol{s}_{n+1}\| - \left\|\boldsymbol{s}_{n+1}^{trial}\right\| + 2\mu\sqrt{\frac{3}{2}}\left[\Delta tA\left(\sqrt{\frac{3}{2}}\right)^{m}\|s_{n+1}\|^{m}t_{n+1}^{n}e^{-Q/RT_{n+1}}\right] \tag{35}$$

The residual is written in terms of a single unknown variable, $\|\boldsymbol{s}_{n+1}\|$ and can be evaulated using a Newton-Raphson iterative scheme. To evaluate the deviatoric stress and the material tangent, the creep strain increment and the derivative of the creep strain increment are required which are respectively given by,

$$\Delta\bar{e}^{cr} = \Delta tA\left(\sqrt{\frac{3}{2}}\right)^{m}\|s_{n+1}\|^{m}t_{n+1}^{n}e^{-Q/RT_{n+1}} \tag{36}$$

$$\frac{\partial\Delta\bar{e}^{cr}}{\partial\|s_{n+1}\|} = \Delta tA\left(\sqrt{\frac{3}{2}}\right)^{m}m\|s_{n+1}\|^{(m-1)}t_{n+1}^{n}e^{-Q/RT_{n+1}} \tag{37}$$

2.5. Combined creep and plasticity

Similarly, the residual to evaluate the plastic multiplier in the case of combined creep and plasticity is given by,

$$R = \sqrt{\frac{2}{3}}\sigma_Y - \left\|\boldsymbol{s}_{n+1}^{trial}\right\| + 2\mu\,\Delta\lambda + 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}_{n+1}^{cr}(\sigma_Y) \tag{38}$$

in which case the deviatoric stress must lie on the yield surface i.e.,

$$\|s_{n+1}\| = \sqrt{\frac{2}{3}}\sigma_Y = \sqrt{\frac{2}{3}}\sigma_{Y0} + \sqrt{\frac{2}{3}}K(\alpha_{n+1}) \tag{39}$$

For this case, the residual can be written in terms of the incremental plastic multiplier which is to be determined.

The creep strain increment in terms of the deviatoric stress is,

$$\Delta\bar{e}_{n+1}^{cr} = \Delta tA\left(\sqrt{\frac{3}{2}}\right)^{m}\|s_{n+1}\|^{m}\,t_{n+1}^{n}e^{-Q/RT_{n+1}} \tag{40}$$

which can be written in terms of the yield stress as,

$$\Delta\bar{e}_{n+1}^{cr} = \Delta tA\sigma_Y^{m}\,t_{n+1}^{n}e^{-Q/RT_{n+1}} \tag{41}$$

To find the solution to the nonlinear equation given by the residual, the derivative of the creep strain increment with respect to the plastic multiplier is required which can be written as,

$$\frac{\partial\Delta\bar{e}_{n+1}^{cr}}{\partial\Delta\lambda} = \Delta tAm\sigma_Y^{(m-1)}\frac{\partial\sigma_Y}{\Delta\lambda}t_{n+1}^{n}e^{-Q/RT_{n+1}} \tag{42}$$

A closed form expression for the derivative of the yield stress can be found if the hardening law is known. For example, considering a linear hardening law, the derivative of the yield stress with respect to the plastic multiplier is,

$$
\begin{aligned}
\frac{\partial \sigma_Y}{\partial \Delta\lambda} &= \frac{\partial}{\partial \Delta\lambda}\left(\sigma_{Y0} + \tilde{K}\alpha_n + \sqrt{\frac{2}{3}}\tilde{K}\Delta\lambda\right) \\
&= \sqrt{\frac{2}{3}}\tilde{K}
\end{aligned}
\tag{43}
$$

The following terms are then required to find the plastic multiplier and the material tangent,

$$
\begin{aligned}
\Delta\bar{e}^{cr} &= \Delta t A \sigma_Y^m \, t_{n+1}^n e^{-Q/RT_{n+1}} \\
&= \Delta t A\left(\sigma_{Y0} + \tilde{K}\alpha_n + \sqrt{\frac{2}{3}}\tilde{K}\Delta\lambda\right)^m t_{n+1}^n e^{-Q/RT_{n+1}}
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\frac{\partial \Delta\bar{e}^{cr}_{n+1}}{\partial \Delta\lambda} &= \Delta t A m \sigma_Y^{(m-1)} \frac{\partial \sigma_Y}{\Delta\lambda} t_{n+1}^n e^{(-Q/RT_{n+1})} \\
&= \Delta t A m \sigma_Y^{(m-1)} \sqrt{\frac{2}{3}}\tilde{K} \, t_{n+1}^n e^{-Q/RT_{n+1}}
\end{aligned}
\tag{45}
$$

3. Consistent Material Tangent

3.1. Combined Plasticity and Creep

The constitutive equation at time step $n+1$ can be written as,

$$
\boldsymbol{\sigma}_{n+1} = 3K\, vol(\boldsymbol{\varepsilon}^{el}_{n+1}) + 2\mu\, \mathrm{dev}(\boldsymbol{\varepsilon}^{el}) \tag{46}
$$

where the elastic strain can be written in terms of the total strain and the inelastic strains. From (5) and (6), (46) can be written as,

$$
\begin{aligned}
\boldsymbol{\sigma}_{n+1} &= 3K\frac{1}{3}\,\mathrm{tr}(\boldsymbol{\varepsilon}^{el}_{n+1})\mathbf{1} + 2\mu\,\mathrm{dev}(\boldsymbol{\varepsilon}^{el}_{n+1}) && (47)\\
&= K\,\mathrm{tr}(\boldsymbol{\varepsilon}^{el}_{n+1})\mathbf{1} + 2\mu\,\mathrm{dev}(\boldsymbol{\varepsilon}^{el}) && (48)\\
&= K\,\mathrm{tr}(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}^{th}_{n+1})\mathbf{1} + 2\mu(\boldsymbol{e}_{n+1} - \boldsymbol{e}^{pl}_{n+1} - \boldsymbol{e}^{cr}_{n+1}) && (49)
\end{aligned}
$$

As in Simo and Hughes [5] (section 3.3.2), the consistent tangent can be obtained by a linearization of the algorithmic constitutive equation (49). Differentiating the expression one obtains,

$$
d\boldsymbol{\sigma}_{n+1} = K\,\mathrm{tr}(d\boldsymbol{\varepsilon}_{n+1} - d\boldsymbol{\varepsilon}^{th}_{n+1})\mathbf{1} + 2\mu(d\boldsymbol{e}_{n+1} - d\boldsymbol{e}^{pl}_{n+1} - d\boldsymbol{e}^{cr}_{n+1}) \tag{50}
$$

where the plastic and creep strains can be written as,

$$
\begin{aligned}
d\boldsymbol{e}^{pl}_{n+1} &= d\boldsymbol{e}^{pl}_{n} + d\Delta\boldsymbol{e}^{pl}_{n+1} && (51)\\
d\boldsymbol{e}^{cr}_{n+1} &= d\boldsymbol{e}^{cr}_{n} + d\Delta\boldsymbol{e}^{cr}_{n+1} && (52)
\end{aligned}
$$

Decomposing (50) into an elastic step and an inelastic correction using Eqs. (51) and (52),

$$d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} - 2\mu\left(d\Delta\boldsymbol{e}^{pl}_{n+1} + d\Delta\boldsymbol{e}^{cr}_{n+1}\right) \tag{53}$$

Rewriting this equation in terms of the total strain,

$$d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \left[\mathbb{C} - 2\mu\frac{\partial\Delta\boldsymbol{e}^{pl}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} - 2\mu\frac{\partial\Delta\boldsymbol{e}^{cr}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right] : d\boldsymbol{\varepsilon}_{n+1} \tag{54}$$

$$= [\mathbb{C} - \boldsymbol{B} - \boldsymbol{A}] : d\boldsymbol{\varepsilon}_{n+1} \tag{55}$$

From (22), the plastic strain increment can be written as,

$$\Delta\boldsymbol{e}^{pl}_{n+1} = \Delta\lambda\boldsymbol{n}_{n+1} \tag{56}$$

Differentiating,

$$\frac{\partial\Delta\boldsymbol{e}^{pl}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} = \frac{\partial(\Delta\lambda\boldsymbol{n}_{n+1})}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{57}$$

$$= \boldsymbol{n}_{n+1} \otimes \frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} + \Delta\lambda\frac{\partial\boldsymbol{n}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{58}$$

Therefore, the second term in the square brackets in (54), $\boldsymbol{B}$ can be written as,

$$\boldsymbol{B} = \boldsymbol{n}_{n+1} \otimes \frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} + \Delta\lambda\frac{\partial\boldsymbol{n}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{59}$$

Similarly from (26),

$$\frac{\partial\Delta\boldsymbol{e}^{cr}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} = \sqrt{\frac{3}{2}}\left(\boldsymbol{n}_{n+1} \otimes \frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} + \Delta\bar{e}^{cr}_{n+1}\frac{\partial\boldsymbol{n}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right) \tag{60}$$

Similar to (59), the third term in the square brackets in (54) can be written as,

$$\boldsymbol{A} = 2\mu\sqrt{\frac{3}{2}}\left(\boldsymbol{n}_{n+1} \otimes \frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} + \Delta\bar{e}^{cr}_{n+1}\frac{\partial\boldsymbol{n}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right) \tag{61}$$

Substituting (59) and (61) in (55) and simplifying,

$$d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \left[\mathbb{C} - 2\mu\frac{\partial\boldsymbol{n}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\left(\Delta\lambda + \sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\right) - 2\mu\boldsymbol{n}_{n+1} \otimes \left(\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} + \sqrt{\frac{3}{2}}\frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}}\right)\right] : d\boldsymbol{\varepsilon}_{n+1} \tag{62}$$

in which matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ that account for the modification of the material stiffness matrix $\mathbb{C}$ due to inelastic effects have to be determined from the trial stress state at time $n+1$. To find these matrices the unit normal tensor $\boldsymbol{n}_{n+1}$, plastic strain increment $\Delta\lambda$, creep strain increment $\Delta\bar{e}^{cr}_{n+1}$ and the respective partial derivatives with respect to the total applied strain $\boldsymbol{\varepsilon}_{n+1}$ need to be evaluated.

3.1.1. Evaluating the unit normal tensor and inelastic strain increments

The unit normal tensor can be evaluated by knowing the trial stress state at $n+1$, which can be determined from the state of the material at n and the total applied strain increment ($\boldsymbol{s}^{trial}_{n+1} = \boldsymbol{s}_n + 2\mu\left(\Delta\boldsymbol{e}_{n+1}\right)$). Both the deviatoric and the trial deviatoric stress, $\boldsymbol{s}_{n+1}$ and $\boldsymbol{s}^{trial}_{n+1}$ can be shown to have the same unit normal tensor $\boldsymbol{n}_{n+1}$ [5], therefore can be determined as,

$$\boldsymbol{n}_{n+1} = \frac{\boldsymbol{s}^{trial}_{n+1}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|} \tag{63}$$

To determine the inelastic strain increments, one starts with the yield function (17), assuming once again a linear isotropic hardening law, can be written in terms of the yield stress in the unhardened state σ_{Y0} and the hardening term $K(\alpha_{n+1})$ as,

$$f_{n+1} = \|\boldsymbol{s}_{n+1}\| - \sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) \tag{64}$$

where α_{n+1} is the accumulated hardening strain at time $n+1$. If the material has yielded, the stress state must lie on the yield surface, or in other words, f_{n+1} must be zero which means,

$$0 = \|\boldsymbol{s}_{n+1}\| - \sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) \tag{65}$$

$$\|\boldsymbol{s}_{n+1}\| = \sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) \tag{66}$$

From the trial deviatoric stress state in (11),

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}^{trial}_{n+1} - 2\mu\left(\Delta\boldsymbol{e}^{pl}_{n+1} + \Delta\boldsymbol{e}^{cr}_{n+1}\right) \tag{67}$$

Substituting in (67) from Eqs. (56) and (26) and contracting with both sides with the unit normal tensor $\boldsymbol{n}_{n+1}$, one obtains,

$$\|\boldsymbol{s}_{n+1}\| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\Delta\lambda - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1} \tag{68}$$

It should also be noted that the magnitude of increment in creep strain here depends on a few variables one of which is the stress state at $n+1$, particularly the von-Mises stress, which indirectly makes it a function of the deviatoric stress at $n+1$. Therefore (68) should ideally be written as,

$$\|\boldsymbol{s}_{n+1}\| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\Delta\lambda - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}(\|\boldsymbol{s}_{n+1}\|) \tag{69}$$

which is to say this is a nonlinear equation in $\|\boldsymbol{s}_{n+1}\|$. Substituting (66) in (68), meaning the deviatoric stress at $n+1$ should lie on the yield surface, one obtains,

$$\sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\Delta\lambda - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\left(\sqrt{\frac{2}{3}}\sigma_{Y0} + \sqrt{\frac{2}{3}}K(\alpha_{n+1})\right) \tag{70}$$

This equation can be solved for the plastic multiplier $\Delta\lambda$ numerically by an iterative Newton-Raphson scheme by writing the residual as,

$$R = \sqrt{\frac{2}{3}}\sigma_{Y0} - \sqrt{\frac{2}{3}}K(\alpha_{n+1}) - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\Delta\lambda + 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\left(\sqrt{\frac{2}{3}}\sigma_{Y0} + \sqrt{\frac{2}{3}}K(\alpha_{n+1})\right) \tag{71}$$

and solving $R \approx 0$, provided an explicit expression is known for the isotropic hardening variable K (e.g. a linear isotropic hardening law can be assumed such that $K(\alpha_{n+1}) = \tilde{K}\alpha_{n+1} = \tilde{K}(\alpha_n + \sqrt{\frac{2}{3}}\Delta\lambda)$) so that the evolution of the yield surface is known in terms of the plastic multiplier. Once the plastic multiplier $\Delta\lambda$ is solved for and the accumulated plastic strain α_{n+1} is known, the creep strain increment $\bar{e}^{cr}_{n+1}$ can be evaluated. Other terms which are needed to evaluate the consistent material tangent are the partial derivatives $\partial \boldsymbol{n}_{n+1}/\partial\boldsymbol{\varepsilon}_{n+1}$, $\partial\Delta\lambda/\partial\boldsymbol{\varepsilon}_{n+1}$ and $\partial\Delta\bar{e}^{cr}_{n+1}/\partial\boldsymbol{\varepsilon}_{n+1}$.

Derivative of the plastic multiplier. The derivative of the plastic multiplier with respect to the total applied strain can be obtained from differentiating the residual in (71),

$$\frac{\partial R}{\partial\boldsymbol{\varepsilon}_{n+1}} = 0 = \sqrt{\frac{2}{3}}\frac{\partial K(\alpha_{n+1})}{\partial\alpha_{n+1}}\frac{\partial\alpha_{n+1}}{\partial\Delta\lambda}\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} - \frac{\partial\left\|\boldsymbol{s}^{trial}_{n+1}\right\|}{\partial\boldsymbol{\varepsilon}_{n+1}} + 2\mu\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} + 2\mu\sqrt{\frac{3}{2}}\frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial\alpha_{n+1}}\frac{\partial\alpha_{n+1}}{\partial\Delta\lambda}\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{72}$$

$$0 = \sqrt{\frac{2}{3}}K'(\alpha_{n+1})\sqrt{\frac{2}{3}}\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} - 2\mu\boldsymbol{n}_{n+1} + 2\mu\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} + 2\mu\sqrt{\frac{3}{2}}\Delta\tilde{\varepsilon}^{cr\prime}_{n+1}\sqrt{\frac{2}{3}}\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{73}$$

where the primes quantities $(.)'$ indicated the derivative with respect to the accumulated plastic strain α_{n+1}. Collecting and rearranging terms, one obtains,

$$\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}}\left(\frac{2}{3}K'(\alpha_{n+1}) + 2\mu\Delta\tilde{\varepsilon}^{cr\prime}_{n+1} + 2\mu\right) = 2\mu\boldsymbol{n}_{n+1} \tag{74}$$

$$\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} = \frac{\boldsymbol{n}_{n+1}}{\left[1 + \Delta\tilde{\varepsilon}^{cr\prime}_{n+1} + \frac{K'(\alpha_{n+1})}{3\mu}\right]} \tag{75}$$

Derivative of the unit normal tensor. From (63), the derivative of the unit normal tensor can be written as,

$$\frac{\partial\boldsymbol{n}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} = \frac{1}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|}\frac{\partial\boldsymbol{s}^{trial}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} - \frac{\boldsymbol{s}^{trial}_{n+1}}{\left\|\boldsymbol{s}^{trial}_{n+1}\right\|^2}\frac{\partial\left\|\boldsymbol{s}^{trial}_{n+1}\right\|}{\partial\alpha_{n+1}}\frac{\partial\alpha_{n+1}}{\partial\Delta\lambda}\otimes\frac{\partial\Delta\lambda}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{76}$$

Knowing that the trial deviatoric stress state is $\boldsymbol{s}^{trial}_{n+1} = \boldsymbol{s}_n + 2\mu\Delta\boldsymbol{e}_{n+1}$, where $\Delta\boldsymbol{e}_{n+1} = \boldsymbol{e}_{n+1} - \boldsymbol{e}_n$ and $\boldsymbol{e}_{n+1} = \mathbb{I}^{dev}\boldsymbol{\varepsilon}_{n+1}$, the derivative can be written as,

$$\frac{\partial\boldsymbol{s}^{trial}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} = \frac{\partial\boldsymbol{s}^{trial}_{n+1}}{\partial\Delta\boldsymbol{e}_{n+1}}\frac{\partial\Delta\boldsymbol{e}_{n+1}}{\partial\boldsymbol{e}_{n+1}}\frac{\partial\boldsymbol{e}_{n+1}}{\partial\boldsymbol{\varepsilon}_{n+1}} \tag{77}$$

$$= 2\mu\,\mathbb{I}^{dev} \tag{78}$$

where $\mathbb{I}^{dev}$ is the deviatoric projection tensor [6], $\mathbb{I}^{dev} = \left[\mathbb{I}^{sym} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right]$

The expression for the derivative of the magnitude of the trial deviatoric stress with respect to the accumulated plastic strain can be obtained by differentiating the expression for the residual in (71),

$$\frac{\partial \left\| \boldsymbol{s}^{trial}_{n+1} \right\|}{\partial \alpha_{n+1}} = \sqrt{\frac{2}{3}} K'(\alpha_{n+1}) + 2\mu \frac{\partial \Delta\lambda}{\partial \alpha_{n+1}} + 2\mu \sqrt{\frac{3}{2}} \Delta\tilde{\varepsilon}^{cr'}_{n+1} \tag{79}$$

$$= \sqrt{\frac{2}{3}} K'(\alpha_{n+1}) + 2\mu \sqrt{\frac{3}{2}} + 2\mu \sqrt{\frac{3}{2}} \Delta\tilde{\varepsilon}^{cr'}_{n+1} \tag{80}$$

where the term $\frac{\partial \Delta\lambda}{\partial \alpha_{n+1}}$ can be evaluated to be $\sqrt{\frac{3}{2}}$, therefore the term $\frac{\partial \alpha_{n+1}}{\partial \Delta\lambda}$ is $\sqrt{\frac{2}{3}}$. Substituting in (76),

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{2\mu\, \mathbb{I}^{dev}}{\left\| \boldsymbol{s}^{trial}_{n+1} \right\|} - \frac{\boldsymbol{s}^{trial}_{n+1}}{\left\| \boldsymbol{s}^{trial}_{n+1} \right\|^2} \left[\frac{2}{3} K'(\alpha_{n+1}) + 2\mu \Delta\tilde{\varepsilon}^{cr'}_{n+1} + 2\mu \right] \otimes \frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{81}$$

$$= \frac{2\mu\, \mathbb{I}^{dev}}{\left\| \boldsymbol{s}^{trial}_{n+1} \right\|} - 2\mu \frac{\boldsymbol{n}_{n+1}}{\left\| \boldsymbol{s}^{trial}_{n+1} \right\|} \left[1 + \Delta\tilde{\varepsilon}^{cr'}_{n+1} + \frac{K'(\alpha_{n+1})}{3\mu} \right] \otimes \frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{82}$$

In light of the expression (75), (82) becomes,

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{2\mu\, \mathbb{I}^{dev}}{\left\| \boldsymbol{s}^{trial}_{n+1} \right\|} - \frac{2\mu}{\left\| \boldsymbol{s}^{trial}_{n+1} \right\|} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \tag{83}$$

$$= \frac{2\mu}{\left\| \boldsymbol{s}^{trial}_{n+1} \right\|} \left(\mathbb{I}^{dev} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \right) \tag{84}$$

Derivative of the creep strain increment. Finally, the derivative of the creep strain increment with respect to the total applied strain can be written as,

$$\frac{\partial \Delta\bar{\varepsilon}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \Delta\bar{\varepsilon}^{cr}_{n+1}}{\partial \alpha_{n+1}} \frac{\partial \alpha_{n+1}}{\partial \Delta\lambda} \frac{\partial \Delta\lambda}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{85}$$

$$= \Delta\tilde{\varepsilon}^{cr'}_{n+1} \sqrt{\frac{2}{3}} \frac{\boldsymbol{n}_{n+1}}{\left[1 + \Delta\tilde{\varepsilon}^{cr'}_{n+1} + \frac{K'(\alpha_{n+1})}{3\mu} \right]} \tag{86}$$

Tangent modulus contd.

Substituting Eqs. (75), (83) and (86) in (62) and rearranging,

$$\begin{aligned} d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \Bigg[\mathbb{C} &- (2\mu)^2 \frac{\mathbb{I}^{dev}}{\left\| \boldsymbol{s}^{trial}_{n+1} \right\|} \left(\Delta\lambda + \sqrt{\frac{3}{2}} \Delta\bar{e}^{cr}_{n+1} \right) \\ &+ (2\mu)^2 \frac{1}{\left\| \boldsymbol{s}^{trial}_{n+1} \right\|} \left(\Delta\lambda + \sqrt{\frac{3}{2}} \Delta\bar{e}^{cr}_{n+1} \right) \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \\ &- \frac{2\mu}{\gamma} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} - \frac{2\mu}{\gamma} \Delta\tilde{\varepsilon}^{cr'}_{n+1} \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \Bigg] : d\boldsymbol{\varepsilon}_{n+1} \end{aligned} \tag{87}$$

or,

$$d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \mathbb{J} : d\boldsymbol{\varepsilon}_{n+1} \tag{88}$$

such that the material tangent is given by,

$$\mathbb{J} = \left[\mathbb{C} - \frac{(2\mu)^2}{\left\|\boldsymbol{s}_{n+1}^{trial}\right\|}\mathbb{I}^{dev}\beta + \left(\frac{(2\mu)^2}{\left\|\boldsymbol{s}_{n+1}^{trial}\right\|}\beta - \frac{2\mu}{\gamma}\Delta\tilde{\varepsilon}_{n+1}^{cr\prime} - \frac{2\mu}{\gamma}\right)\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right] \tag{89}$$

where, $\gamma = \left[1 + \Delta\tilde{\varepsilon}_{n+1}^{cr\prime} + \frac{K'(\alpha_{n+1})}{3\mu}\right]$ and $\beta = \left(\Delta\lambda + \sqrt{\frac{3}{2}}\Delta\bar{e}_{n+1}^{cr}\right)$.
As a sanity check, it can be verified that in the absence of creep effects, the material tangent reduces to,

$$\mathbb{J}^{ep} = \left[\mathbb{C} - \frac{(2\mu)^2}{\left\|\boldsymbol{s}_{n+1}^{trial}\right\|}\mathbb{I}^{dev}\Delta\lambda + \left(\frac{(2\mu)^2}{\left\|\boldsymbol{s}_{n+1}^{trial}\right\|}\Delta\lambda - \frac{2\mu}{\left[1 + \frac{K'(\alpha_{n+1})}{3\mu}\right]}\right)\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right] \tag{90}$$

which is the same as the consistent elastoplastic tangent given in Simo and Hughes [5], pg. 124.,

3.2. *Pure Creep*

The previous section details the material tangent for combined creep and plasticity behavior where it is assumed explicitly that the material has yielded, therefore the stress state at the current time step $n+1$ lies on the yield surface. However, if the material has not yielded, the creep strain at $n+1$ depends on the stress state at $n+1$ which is yet to be determined. From the Eqn. (53), assuming that there is no plasticity,

$$d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} - 2\mu\left(d\Delta\boldsymbol{e}_{n+1}^{cr}\right) \tag{91}$$

From Eqn. (26),

$$\Delta\boldsymbol{e}_{n+1}^{cr} = \Delta\bar{e}_{n+1}^{cr}\left(\tilde{\sigma}_{n+1}, T_{n+1}, t_{n+1}\right)\sqrt{\frac{3}{2}}\boldsymbol{n}_{n+1} \tag{92}$$

Differentiating this expression,

$$d\Delta\boldsymbol{e}_{n+1}^{cr} = d\left(\Delta\bar{e}_{n+1}^{cr}\left(\tilde{\sigma}_{n+1}, T_{n+1}, t_{n+1}\right)\sqrt{\frac{3}{2}}\boldsymbol{n}_{n+1}\right) \tag{93}$$

$$= \Delta\bar{e}_{n+1}^{cr}\sqrt{\frac{3}{2}}d\boldsymbol{n}_{n+1} + \sqrt{\frac{3}{2}}\boldsymbol{n}_{n+1}d\Delta\bar{e}_{n+1}^{cr} \tag{94}$$

Substituting Eqn. (94) in Eqn. (91),

$$d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} - 2\mu\left(\Delta\bar{e}_{n+1}^{cr}\sqrt{\frac{3}{2}}d\boldsymbol{n}_{n+1} + \sqrt{\frac{3}{2}}\boldsymbol{n}_{n+1}d\Delta\bar{e}_{n+1}^{cr}\right) \tag{95}$$

Writing in terms of the total strain,

$$d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \left[\mathbb{C} - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} - 2\mu\sqrt{\frac{3}{2}}\boldsymbol{n}_{n+1} \otimes \frac{\partial \Delta\bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right] : d\boldsymbol{\varepsilon}_{n+1} \tag{96}$$

The partial derivative of the unit normal tensor with respect to the total strain can be written as,

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{s}_{n+1}}\frac{\partial \boldsymbol{s}_{n+1}}{\partial \boldsymbol{\sigma}_{n+1}}\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{97}$$

where from Eqn. 3.3.9 in Simo and Hughes [5], (see also appendix),

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{s}_{n+1}} = \frac{1}{||\boldsymbol{s}_{n+1}||}\left[\mathbb{I}^{sym} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right] \tag{98}$$

$$\frac{\partial \boldsymbol{s}_{n+1}}{\partial \boldsymbol{\sigma}_{n+1}} = \mathbb{I}^{dev} \tag{99}$$

where, $\mathbb{I}^{dev}$ is the deviatoric projection tensor given by,

$$\mathbb{I}^{dev} = \left[\mathbb{I}^{sym} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}\right] \tag{100}$$

such that,

$$\boldsymbol{s}_{n+1} = \mathbb{I}^{dev} : \boldsymbol{\sigma}_{n+1} \tag{101}$$

Substituting in Eqn. (97),

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{1}{||\boldsymbol{s}_{n+1}||}\left[\mathbb{I}^{sym} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right]\mathbb{I}^{dev}\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{102}$$

which can be simplified to,

$$\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{1}{||\boldsymbol{s}_{n+1}||}\left[\mathbb{I}^{dev} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right]\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{103}$$

Similarly,

$$\frac{\partial \Delta\bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \Delta\bar{e}^{cr}_{n+1}}{\partial ||\boldsymbol{s}_{n+1}||}\frac{\partial ||\boldsymbol{s}_{n+1}||}{\partial \boldsymbol{\sigma}_{n+1}}\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{104}$$

where $\frac{\partial ||\boldsymbol{s}_{n+1}||}{\partial \boldsymbol{\sigma}_{n+1}} = \boldsymbol{n}_{n+1}$ from Eqn. (21) and $\frac{\partial \Delta\bar{e}^{cr}_{n+1}}{\partial ||\boldsymbol{s}_{n+1}||}$ is yet to be determined. Substituting in Eqn. (104),

$$\frac{\partial \Delta\bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \Delta\bar{e}^{cr}_{n+1}}{\partial ||\boldsymbol{s}_{n+1}||}\boldsymbol{n}_{n+1}\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \tag{105}$$

Substituting for $\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}$ and $\frac{\partial \Delta\bar{e}^{cr}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}$ in Eqn. (96) and rearranging,

$$\begin{aligned} d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} &- \left[2\mu\sqrt{\frac{3}{2}}\frac{\Delta\bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||}\left[\mathbb{I}^{dev} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right]\right] : d\boldsymbol{\sigma}_{n+1} \\ &- \left[2\mu\sqrt{\frac{3}{2}}\frac{\partial \Delta\bar{e}^{cr}_{n+1}}{\partial ||\boldsymbol{s}_{n+1}||}\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right] : d\boldsymbol{\sigma}_{n+1} \end{aligned} \tag{106}$$

moving the second and third term from the right hand side to the left,

$$\left[\mathbb{I}^{sym} + 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\frac{1}{||\boldsymbol{s}_{n+1}||}\left(\mathbb{I}^{dev} - \boldsymbol{n}_{n+1}\otimes\boldsymbol{n}_{n+1}\right) + 2\mu\sqrt{\frac{3}{2}}\frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial||\boldsymbol{s}_{n+1}||}\boldsymbol{n}_{n+1}\otimes\boldsymbol{n}_{n+1}\right] : d\boldsymbol{\sigma}_{\boldsymbol{n+1}} \\ = \mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} \tag{107}$$

where $\mathbb{I}^{sym}$ is the symmetric fourth order identity tensor. If the term in the square brackets on the left hand side is written at $\tilde{\boldsymbol{A}}$, then Eqn. (107) can be written as,

$$\tilde{\boldsymbol{A}} : d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} \tag{108}$$

and the material tangent can be written as,

$$d\boldsymbol{\sigma}_{\boldsymbol{n+1}} = \tilde{\boldsymbol{A}}^{-1}\mathbb{C} : d\boldsymbol{\varepsilon}_{n+1} \tag{109}$$

where,

$$\tilde{\boldsymbol{A}} = \left[\mathbb{I}^{sym} + 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}\frac{1}{||\boldsymbol{s}_{n+1}||}\left(\mathbb{I}^{dev} - \boldsymbol{n}_{n+1}\otimes\boldsymbol{n}_{n+1}\right) + 2\mu\sqrt{\frac{3}{2}}\frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial||\boldsymbol{s}_{n+1}||}\boldsymbol{n}_{n+1}\otimes\boldsymbol{n}_{n+1}\right] \tag{110}$$

$$= \left[\mathbb{I}^{sym} + 2\mu\sqrt{\frac{3}{2}}\frac{\Delta\bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||}\mathbb{I}^{dev} + 2\mu\sqrt{\frac{3}{2}}\left(-\frac{\Delta\bar{e}^{cr}_{n+1}}{||\boldsymbol{s}_{n+1}||} + \frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial||\boldsymbol{s}_{n+1}||}\right)\boldsymbol{n}_{n+1}\otimes\boldsymbol{n}_{n+1}\right] \tag{111}$$

Here the unit normal tensor $\boldsymbol{n}_{n+1}$ can be obtained directly from the trial deviatoric stress, however the magnitude of the deviatoric stress, the creep strain increment and derivative of the creep strain increment, i.e. $||\boldsymbol{s}_{n+1}||$, $\Delta\bar{e}^{cr}_{n+1}$ and $\frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial||\boldsymbol{s}_{n+1}||}$ are yet to be determined. Similar to Eqn. (68), the magnitude of the deviatoric stress can be written as,

$$||\boldsymbol{s}_{n+1}|| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1} \tag{112}$$

where the creep strain increment $\Delta\bar{e}^{cr}_{n+1}$ is some function of the current stress state $\boldsymbol{\sigma}_{n+1}$ and furthermore is generally a function of the current von-Mises stress state or indirectly, the deviatoric stress state,

$$||\boldsymbol{s}_{n+1}|| = \left\|\boldsymbol{s}^{trial}_{n+1}\right\| - 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}(||\boldsymbol{s}_{n+1}||) \tag{113}$$

Therefore, this is in general, a nonlinear equation in $||\boldsymbol{s}_{n+1}||$ which can be solved by a Newton-Raphson iterative scheme. If the residual is written as,

$$R = ||\boldsymbol{s}_{n+1}|| - \left\|\boldsymbol{s}^{trial}_{n+1}\right\| + 2\mu\sqrt{\frac{3}{2}}\Delta\bar{e}^{cr}_{n+1}(||\boldsymbol{s}_{n+1}||) \tag{114}$$

solving for $R \approx 0$, the quantities $||\boldsymbol{s}_{n+1}||$, $\Delta\bar{e}^{cr}_{n+1}$ and $\frac{\partial\Delta\bar{e}^{cr}_{n+1}}{\partial||\boldsymbol{s}_{n+1}||}$ can be determined provided an explicit expression for the creep strain increment in terms of the magnitude of deviatoric stress is known. Substituting these quantities in Eqn. (110), the material tangent can be obtained from Eqn. (109).

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References

[1] Aydin Karahan. *Modelling of thermo-mechanical and irradiation behavior of metallic and oxide fuels for sodium fast reactors*. PhD thesis, Massachusetts Institute of Technology, 2009.

[2] RL Williamson, JD Hales, SR Novascone, MR Tonks, DR Gaston, CJ Permann, D Andrs, and RC Martineau. Multidimensional multiphysics simulation of nuclear fuel behavior. *Journal of Nuclear Materials*, 423(1-3):149–163, 2012.

[3] Miloš Kojić and Klaus-Jürgen Bathe. The effective-stress-functionalgorithm for thermo-elasto-plasticity and creep. *International journal for numerical methods in engineering*, 24(8):1509–1532, 1987.

[4] Surya R Kalidindi, Curt A Bronkhorst, and Lallit Anand. Crystallographic texture evolution in bulk deformation processing of fcc metals. *Journal of the Mechanics and Physics of Solids*, 40(3):537–569, 1992.

[5] Juan C Simo and Thomas JR Hughes. *Computational inelasticity*, volume 7. Springer Science & Business Media, 2006.

[6] Milan Jirásek. Basic concepts and equations of solid mechanics. *Revue européenne de génie civil*, 11(7-8):879–892, 2007.

Appendix A. Appendices

Appendix A.1. Normal tensor

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial(\sqrt{2J_2})}{\partial \boldsymbol{\sigma}} \tag{A.1}$$

$$= \frac{1}{2\sqrt{2J_2}} \frac{\partial(2J_2)}{\partial \boldsymbol{\sigma}} \tag{A.2}$$

$$= \frac{1}{\sqrt{\boldsymbol{s} : \boldsymbol{s}}} \frac{\partial J_2}{\partial \boldsymbol{\sigma}} \tag{A.3}$$

J_2, the second invariant of the deviatoric stress can be written in terms of the invariants of the stress tensor as,

$$J_2 = \frac{1}{3} I_1^2 - I_2 \tag{A.4}$$

The gradient of J_2 can be written as

$$\begin{aligned} \frac{\partial J_2}{\partial \boldsymbol{\sigma}} &= \frac{1}{3} 2 I_1 \frac{\partial I_1}{\partial \boldsymbol{\sigma}} - \frac{\partial I_2}{\partial \boldsymbol{\sigma}} \\ &= \frac{2}{3} I_1 \mathbf{1} - (I_1 \mathbf{1} - \boldsymbol{\sigma}^T) \\ &= \boldsymbol{\sigma} - \left(I_1 - \frac{2}{3} I_1 \right) \mathbf{1} \\ &= \boldsymbol{\sigma} - \frac{1}{3} \mathrm{tr}(\boldsymbol{\sigma}) \mathbf{1} \\ &= \boldsymbol{s} \end{aligned} \tag{A.5}$$

Substituting (A.5) in (A.3), the gradient of f with respect to $\boldsymbol{\sigma}$ is obtained as,

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\boldsymbol{s}}{\sqrt{\boldsymbol{s} : \boldsymbol{s}}} \tag{A.6}$$

$$= \boldsymbol{n} \tag{A.7}$$

Appendix A.2. Derivative of the unit normal tensor with the deviatoric stress

The unit normal tensor is given by

$$\boldsymbol{n} = \frac{\boldsymbol{s}}{\sqrt{\boldsymbol{s} : \boldsymbol{s}}} \tag{A.8}$$

Taking the derivative with respect to the deviatoric stress tensor and changing to index notation

$$\frac{\partial n_{ij}}{\partial s_{kl}} = \frac{\sqrt{\boldsymbol{s} : \boldsymbol{s}} \frac{\partial s_{ij}}{\partial s_{kl}} - s_{ij} \frac{\partial \sqrt{\boldsymbol{s}:\boldsymbol{s}}}{\partial s_{kl}}}{(\sqrt{\boldsymbol{s} : \boldsymbol{s}})^2} \tag{A.9}$$

where the first derivative in the numerator is the symmetric fourth order identity tensor,

$$\frac{\partial s_{ij}}{\partial s_{kl}} = I^{sym}_{ijkl} \tag{A.10}$$

To evaluate the second derivative in the numerator, first write

$$\sqrt{\boldsymbol{s} : \boldsymbol{s}} = \sqrt{2J_2} \tag{A.11}$$

Which implies,

$$\begin{aligned}
\frac{\partial \sqrt{\boldsymbol{s} : \boldsymbol{s}}}{\partial s_{kl}} &= \frac{\partial \sqrt{2J_2}}{\partial s_{kl}} \\
&= \frac{\partial \sqrt{2J_2}}{\partial \sigma_{mn}} \frac{\partial \sigma_{mn}}{\partial s_{kl}} \\
&= \frac{\sqrt{2}}{2\sqrt{J_2}} \frac{\partial J_2}{\partial \sigma_{mn}} I^{dev}_{mnkl} \\
&= \frac{1}{\sqrt{2J_2}} s_{mn} I^{dev}_{mnkl} \\
&= \frac{1}{||s||} s_{kl} \\
&= n_{kl}
\end{aligned} \tag{A.12}$$

Substituting in (A.9)

$$\begin{aligned}
\frac{\partial n_{ij}}{\partial s_{kl}} &= \frac{\sqrt{\boldsymbol{s} : \boldsymbol{s}} I^{sym}_{ijkl} - s_{ij} n_{kl}}{(\sqrt{\boldsymbol{s} : \boldsymbol{s}})^2} \\
&= \frac{||s|| I^{sym}_{ijkl} - s_{ij} n_{kl}}{||s||^2} \\
&= \frac{I^{sym}_{ijkl} - n_{ij} n_{kl}}{||s||} \\
&= \frac{1}{||s||} \left(I^{sym}_{ijkl} - n_{ij} n_{kl} \right)
\end{aligned} \tag{A.13}$$

Or in tensor notation, this can be written as

$$\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{s}} = \frac{1}{||s||} (\mathbb{I}^{sym} - \boldsymbol{n} \otimes \boldsymbol{n}) \tag{A.14}$$